Buckling of Cylindrical Shells with Smeared-Out and Discrete Orthogonal Stiffeners

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The buckling equations for the orthogonally stiffened cylindrical shells under uniform axial compression and external pressure and with classical simply supported boundary conditions are formulated by treating the stiffeners as discrete elements. By assuming identical and equally spaced stringers and identical and equally spaced rings, the buckling equations can be uncoupled into several sets of simpler and manageable equations for the symmetric and antisymmetric longitudinal modes and symmetric and antisymmetric circumferential modes. The uncoupled submatrices are further reduced by partitioning and substitution. Effort is made to preserve the sparseness of the matrices in order to use a special compact storage scheme. A method to compute the minimum eigenvalue for a large general eigenvalue problem, the Ritz iteration method combined with Chebyshev procedure, is developed and its accuracies are evaluated. Examples are performed and results are compared to other computational and experimental results available.

Nomenclature

A	= cross-sectional area
a_1,a_2	= spacings between longitudinal and circumferential smeared-out stiffeners, respectively
d_1,d_2	= depths of the longitudinal and circumferential smeared-out stiffeners, respectively
d_s,d_r	= depths of the discrete stringers and rings, respectively
E_s,E_r	= Young's moduli of the discrete stringers and rings, respectively
h	= thickness of the shell skin
h_{e1},h_{e2}	= equivalent thicknesses of the smeared-out longitudinal and circumferential stiffeners defined, respectively, as d_1t_1/a_1 and $d_2t_2(1/a_2-1/L)$
h_{es},h_{er}	= equivalent thicknesses of the discrete stringers and rings defined, respectively, as $A_s/2\pi r$ and A_r/L
J	$=\sum_{k=1}^{n}[1-(192t/\pi^5 d)\tanh(\pi d/2t)]t^3d/3$ torsional constant of the stiffener[22]
L	= length of the cylindrical shell
m,n	= longitudinal half-wave and circumferential full-wave numbers, respectively
n_s,n_r	= number of discrete stringers and rings, respectively
N_x	= uniform axial buckling load defined as $P_x/2\pi r$
P	= external buckling pressure
P_x	= uniform axial buckling load
r	= mean radius of the cylindrical shell skin
t_1,t_2	= widths of the longitudinal and circumferential smeared-out stiffeners, respectively
t_s,t_r	= widths of the discrete stringer and ring, respectively
u,v,w	= displacement components in x, θ, and z directions, respectively
U_{mn},V_{mn},W_{mn}	= amplitudes in the displacement functions
x,θ,z	= cylindrical coordinates shown in Fig. 1
Z	$=L^2(1-\nu^2)^{1/2}/hr$, geometric parameter
$\epsilon_1\ \epsilon_2\ \epsilon_{12}$	= direct and shearing strain components
$\epsilon_x,\epsilon_\theta$	= longitudinal and circumferential strains in the shell skin, respectively
$\epsilon_{x1},\epsilon_{\theta 2}$	= strains of the longitudinal and circumferential smeared-out stiffeners, respectively
$\epsilon_{xs},\epsilon_{\theta r}$	= strains of the discrete stringers and rings, respectively
ν	= Poisson's ratio
ρ	= weight density
σ	= stresses with subscripts 1, 2, 12, x, θ, $x1$, $\theta 2$, xs, and θr having the same meanings as those used for strains ϵ
$[x]$	= Gaussian symbol defined as the integer not greater than x and with the same sign as x
$i_1\equiv i_2(\mathrm{mod}\ i_3)$	$=i_1$ being congruent to i_2 modulo i_3, defined as $i_2-[i_2/i_3]i_3$

Introduction

SINCE van der Neut[1] demonstrated the influence of eccentricity of stiffeners on the buckling strength of stiffened cylindrical shell in 1947, numerous papers[2-10] on the buckling analysis of stiffened cylindrical shells using the smeared-out technique were published. Such studies were motivated by the structural efficiency and practical advantages of the wall stiffening. Reference 3 showed that when the smeared-out method was used for cylinders with large number of stiffeners, good correlation between experimental and theoretical results in buckling load was obtained. However, this may not be the case for cylinders with a small number of stiffeners.

MacNeal et al.[11] treated ring-stiffened cylindrical shells by considering rings as discrete elements. Wang and Lin[12] and Singer and Haftka[13] analyzed the buckling of cylindrical shells with discrete stringers and showed the differences in results between the smeared-out and the discrete methods.

Egle and Sewall[14] derived the frequency equations for the cylindrical shells with discrete stringers and rings. Useful

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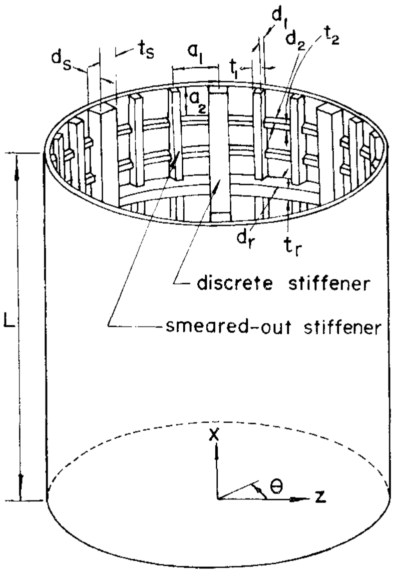

Fig. 1 Cylindrical shell with smeared-out and discrete orthogonal stiffeners.

information on circumferential modes, which are in common for both vibration and buckling problems, was given. Their numerical results were limited to the study of the effect of discrete stringers on natural frequencies. McDonald[15] solved a problem on the free vibration of stiffened cylindrical shells by considering rings as discrete elements.

Wang[16] performed the deformation and stress calculations for orthogonally stiffened cylinders under internal pressure by treating the stiffeners as separate elements. Soong[17] analyzed the general instability of orthotropic stiffened cylinders with intermittently attached stiffeners by considering the discreteness of the stiffeners. The buckling patterns assumed were symmetric circumferential and symmetric longitudinal modes. The results calculated on the basis of those displacement functions for the cylindrical shells with many stiffeners agreed well with experimental data and smeared-out calculations.

In this study, the discreteness of the stiffeners is considered in the buckling analysis of orthogonally stiffened cylinders with classical simply supported edge conditions. It is shown that, when all of the stringers are identical and equally spaced, the buckling equations can be reduced to two sets of equations: one set yields the antisymmetric circumferential modes and the other yields the symmetric circumferential modes. It is also shown that, when all of the rings are identical and equally spaced, the buckling equations can be reduced to two sets of equations: one set yields the antisymmetric longitudinal modes and the other yields the symmetric longitudinal modes. The uncoupled submatrices are further reduced by partitioning and substitution. Effort is made to preserve the sparseness of the matrices in order to save computing time and storage by using a special compact storage scheme.[18]

The large sets of general eigenvalue buckling equations are solved by the Ritz iteration method combined with Chebyshev procedure.[19] The accuracy and efficiency of the method are evaluated through an example. Examples of buckling predictions for discretely stiffened cylinders are performed and results are compared with available analytical and experimental solutions.[20]

Fundamental Equations

Basic Assumptions

For the shell skin, it is assumed that the Kirchhoff and Love assumptions for thin elastic shell hold. For the stiffeners, it is assumed that: 1) the shell is integrally stiffened by longitudinal stringers and circumferential rings (both smeared-out and discrete types) and 2) the smeared-out stiffeners are thin and deep such that the uniaxial stress condition prevails and the torsion stiffness is neglected.

Total Potential Energy Expressions

The total potential energy for the system is defined as

$$\Pi = U_1 + U_2 + U_3 + U_4 - W_1 - W_2 - W_3 \tag{1}$$

where U_1 is the strain energy for the shell skin; U_2 is the strain energy for the smeared-out orthogonal stiffeners; U_3 and U_4 are the strain energies for the discrete stringers and rings, respectively; W_1, W_2, and W_3 are the potential energies due to the middle-surface load for the shell with the smeared-out stiffeners, the discrete stringers, and the discrete rings, respectively.

The strain energy expressions written in terms of strains are available collectively in, for example, Refs. 2, 10, and 14. These expressions can be derived in terms of displacement by using the following strain-displacement relations

$$\epsilon_1 = u_{,x} - z w_{,xx} \tag{2a}$$

$$\epsilon_2 = \frac{1}{r} v_{,\theta} - \frac{w}{r} - \frac{z}{r^2}(v_{,\theta} + w_{,\theta\theta}) \tag{2b}$$

$$\epsilon_{12} = v_{,x} + \frac{1}{r} u_{,\theta} - 2\frac{z}{r}\left(w_{,x\theta} + \frac{1}{2} v_{,x}\right) \tag{2c}$$

It is noted that the terms $zv_{,\theta}/r^2$ and $zv_{,x}/r$ are neglected if Donnell-type theory is used. The strain energy vs displacement expressions based on Eq. (2) were derived in Ref. 18.

The potential energy vs displacement expressions take the following forms:

$$W_1 = \int_0^L \int_0^{2\pi} \left\{ \left(u_{,x} + \frac{1}{2} v^2_{,x} + \frac{1}{2} w^2_{,x}\right)(\sigma_x h + \sigma_{x1} h_{e1}) + \left(\frac{1}{r} v_{,\theta} - \frac{1}{r} w_{,r} + \frac{1}{2}\frac{1}{r^2} u^2_{,\theta} + \frac{1}{2}\frac{1}{r^2} w^2_{,\theta}\right) \times (\sigma_\theta h + \sigma_{\theta 2} h_{e2}) \right\} r\mathrm{d}\theta \mathrm{d}x \tag{3}$$

$$W_2 = \sum_{k=1}^{n_s} \int_0^L \sigma_{xs} A_{s_k} \left(u_{,x} + \frac{1}{2} w^2_{,x}\right)_{\theta=\theta_k} \mathrm{d}x \tag{4}$$

$$W_3 = \sum_{k=1}^{n_r} \int_0^{2\pi} \sigma_{\theta r} A_{r_k} \left(\frac{1}{r} v_{,\theta} - \frac{w}{r} + \frac{1}{2}\frac{1}{r^2} u^2_{,\theta} + \frac{1}{2}\frac{1}{r^2} w^2_{,\theta}\right)_{x=x_k} r\mathrm{d}\theta \tag{5}$$

Displacement Functions

The displacement functions which satisfy the classical simply supported boundary conditions that $w = M_x = N_x = v = 0$ at $x = 0$ and L may be represented as

$$u = \sum_{m=1}^{k'_m} \frac{U''_{mo}}{2} \cos\frac{m\pi x}{L} + \sum_{m=1}^{k'_m} \sum_{n=1}^{k'_n - 1} (U'_{mn} \sin n\theta + U''_{mn} \cos n\theta) \cos\frac{m\pi x}{L} \tag{6a}$$

$$v = \sum_{m=1}^{k'_m} \frac{V'_{mo}}{2} \sin\frac{m\pi x}{L} + \sum_{m=1}^{k'_m} \sum_{n=1}^{k'_n - 1} (V'_{mn} \cos n\theta + V''_{mn} \sin n\theta) \sin\frac{m\pi x}{L} \tag{6b}$$

$$w=\sum_{m=1}^{k_m}\frac{W''_{mo}}{2}\sin\frac{m\pi x}{L}+\sum_{m=1}^{k_m}\sum_{n=1}^{k_n-1}(W'_{mn}\sin n\theta+W''_{mn}\cos n\theta)\sin\frac{m\pi x}{L}\tag{6c}$$

Equilibrium Equations for Buckling Analysis

Substituting the displacement functions of Eqs. (6) into the expressions for the total potential energy[18] and then using the minimum potential energy principle, the equations of equilibrium are obtained as

$$\left[\begin{bmatrix}\tilde C'_{11} & & & & \text{Symmetric} & \\ \tilde C'_{21} & \tilde C'_{22} & & & & \\ \tilde C'_{31} & \tilde C'_{32} & \tilde C'_{33} & & & \\ \tilde C'''_{11} & \tilde C'''_{12} & \tilde C'''_{13} & \tilde C''_{11} & & \\ \tilde C'''_{21} & \tilde C'''_{22} & \tilde C'''_{23} & \tilde C''_{21} & \tilde C''_{22} & \\ \tilde C'''_{31} & \tilde C'''_{32} & \tilde C'''_{33} & \tilde C''_{31} & \tilde C''_{32} & \tilde C''_{33}\end{bmatrix} - \begin{bmatrix}\tilde 0 & & & & \text{Symmetric} & \\ \tilde 0 & \tilde 0 & & & & \\ \tilde 0 & \tilde 0 & \tilde D'_{33} & & & \\ \tilde 0 & \tilde 0 & \tilde 0 & \tilde 0 & & \\ \tilde 0 & \tilde 0 & \tilde 0 & \tilde 0 & \tilde 0 & \\ \tilde 0 & \tilde 0 & \tilde D'''_{33} & \tilde 0 & \tilde 0 & \tilde D''_{33}\end{bmatrix}\right]\begin{Bmatrix}\tilde U'\\ \tilde V'\\ \tilde W'\\ \tilde U''\\ \tilde V''\\ \tilde W''\end{Bmatrix}=\begin{Bmatrix}\tilde 0\\ \tilde 0\\ \tilde 0\\ \tilde 0\\ \tilde 0\\ \tilde 0\end{Bmatrix}\tag{7}$$

where the coefficients in the two symmetric matrices are defined explicitly in Ref. 18.

The order of the matrices in Eqs. (7) is $(4k'_m k'_n + 2k_m k_n)$. The submatrices $\tilde C'''$ and $\tilde D'''$ are functions of the properties of the discrete stringers only. If there are no stringers, Eqs. (7) reduce to two sets of equations which result in the same buckling load but two circumferential modes with 90-deg phase difference.

The uniform axial load and uniform external pressure are considered here. They are related to the internal stresses by using the assumption of either uniform stresses or uniform strains in the system. The stresses based on the uniform stress assumption are

$$\sigma_x = N_x/(h+h_{es}+h_{e1}) \text{ and } \sigma_\theta = pr/(h+h_{er}+h_{e2})\tag{8}$$

The stresses based on the uniform strain assumptions are

$$\sigma_x=\{N_x(H_2+H_r+1)+\nu pr(H_1+H_s)\}/hH\tag{9a}$$

$$\sigma_\theta=\{\nu N_x(H_2+H_r)+pr(H_1+H_s+1)\}/hH\tag{9b}$$

$$\sigma_{x1}=[N_x\{(H_2+H_r)(1-\nu^2)+1\}-\nu pr]/hH\tag{9c}$$

$$\sigma_{\theta 2}=[-\nu N_x+pr\{(H_1+H_s)(1-\nu^2)+1\}]/hH\tag{9d}$$

$$\sigma_{xs}=E_s[N_x\{(H_2+H_r)(1-\nu^2)+1\}-\nu pr]/EhH\tag{9e}$$

$$\sigma_{\theta r}=E_r[-\nu N_x+pr\{H_1+H_s)(1-\nu^2)+1\}]/EhH\tag{9f}$$

where $H=(H_1+H_s+1)(H_2+H_r+1)-\nu^2(H_1+H_s)(H_2+H_r)$; $H_1=h_{e1}/h$; $H_2=h_{e2}/h$; $H_s=E_s h_{es}/Eh$; and $H_r=E_r h_{er}/Eh$. The axial load and external pressure have the relation that $N_x=pr/2$.

Equally Spaced and Identical Discrete Stringers and Rings

If the stringers are identical in size, the matrix element with triple primes in Eqs. (7) include a common term,

$$L_{n_in_j}=\sum_{k=1}^{n_s}\sin n_i\theta_k\cos n_j\theta_k\tag{10}$$

For equally spaced stringers, $\theta_k=2\pi k/n_s$ and Eq. (10) becomes

$$L_{n_in_j}=\frac{1}{2}\sum_{k=1}^{n_s}\sin\frac{2\pi(n_i-n_j)}{n_s}k+\frac{1}{2}\sum_{k=1}^{n_s}\times\sin\frac{2\pi(n_i+n_j)}{n_s}k=L'_{n_in_j}+L''_{n_in_j}\tag{11}$$

where $L'_{n_in_j}=L''_{n_in_j}=0$ if $n_i-n_j\equiv 0 \pmod{n_s}$ and $n_i+n_j\equiv 0 \pmod{n_s}$. However, if $n_i-n_j\not\equiv 0 \pmod{n_s}$ and $n_i+n_j\not\equiv 0 \pmod{n_s}$, it can be derived that[21]

$$L'_{n_in_j}=\frac{1}{2}\sin\frac{\pi(n_i-n_j)(n_s+1)}{n_s}\sin(n_i-n_j)\pi\operatorname{cosec}\frac{n_i-n_j}{n_s}\pi=0\tag{12a}$$

$$L''_{n_in_j}=\frac{1}{2}\sin\frac{\pi(n_i+n_j)(n_s+1)}{n_s}\sin(n_i+n_j)\pi\operatorname{cosec}\frac{n_i+n_j}{n_s}\pi=0\tag{12b}$$

Thus, it is shown that for the case of identical and equally spaced stringers, the submatrix elements with triple primes in Eqs. (7) vanish. Equations (7) are thus uncoupled into two sets of equations. One set is related to the amplitudes U', V', and W' and yields the antisymmetric circumferential buckling modes. The other is related to the amplitudes U'', V'', and W'' and yields the symmetric circumferential buckling modes.

The elements in the submatrices with single and double primes contain two common terms

$$M_{n_in_j}=\sum_{k=1}^{n_s}\cos n_i\theta_k\cos n_j\theta_k$$
$$N_{n_in_j}=\sum_{k=1}^{n_s}\sin n_i\theta_k\sin n_j\theta_k\tag{13}$$

Using the similar procedures as used in deriving Eqs. (11) and (12), Eq. (13) becomes

$$M_{n_in_j}=M'_{n_in_j}+M''_{n_in_j} \text{ and } N_{n_in_j}=N'_{n_in_j}-N''_{n_in_j}\tag{14}$$

where

$$M'_{n_in_j}=N'_{n_in_j}=0 \text{ if } n_i-n_j\not\equiv 0 \pmod{n_s}\tag{15a}$$

$$M'_{n_in_j}=N'_{n_in_j}=n_s/2 \text{ if } n_i-n_j\equiv 0 \pmod{n_s}\tag{15b}$$

$$M''_{n_in_j}=N''_{n_in_j}=0 \text{ if } n_i+n_j\not\equiv 0 \pmod{n_s}\tag{15c}$$

$$M''_{n_in_j}=N''_{n_in_j}=n_s/2 \text{ if } n_i+n_j\equiv 0 \pmod{n_s}\tag{15d}$$

For identical and equally spaced rings, the buckling Eqs. (7) include the following terms,

$$Q_{m_im_j}=\sum_{k=1}^{n_r}\cos\frac{m_i\pi X_k}{L}\cos\frac{m_j\pi X_k}{L}\tag{16a}$$

$$R_{m_im_j}=\sum_{k=1}^{n_r}\sin\frac{m_i\pi X_k}{L}\sin\frac{m_j\pi X_k}{L}\tag{16b}$$

with $X_k = kL/(n_r+1)$. Similar derivations give

$$Q_{m_i m_j} = Q'_{m_i m_j} + Q''_{m_i m_j} \quad \text{and} \quad R_{m_i m_j} = Q'_{m_i m_j} - Q''_{m_i m_j} \tag{17}$$

where

$$Q'_{m_i m_j} = \frac{1}{2}\cos\frac{\pi}{2}(m_i - m_j)\sin\frac{\pi}{2}\frac{n_r(m_i-m_j)}{n_r+1} \times \operatorname{cosec}\frac{\pi}{2}\frac{(m_i-m_j)}{(n_r+1)} \quad \text{if } m_i - m_j \not\equiv 0 (\text{mod } 2n_r+2) \tag{18}$$

$$Q'_{m_i m_j} = n_r/2 \quad \text{if } m_i - m_j \equiv 0(\text{mod } 2n_r+2) \tag{19}$$

$$Q''_{m_i m_j} = \frac{1}{2}\cos\frac{\pi}{2}(m_i + m_j)\sin\frac{\pi}{2}\frac{n_r(m_i-m_j)}{n_r+1} \times \operatorname{cosec}\frac{\pi}{2}\frac{(m_i-m_j)}{(n_r+1)} \quad \text{if } m_i + m_j \not\equiv 0 (\text{mod } 2n_r+2) \tag{20}$$

$$Q''_{m_i m_j} = n_r/2 \quad \text{if } m_i + m_j \equiv 0(\text{mod } 2n_r+2) \tag{21}$$

Further examination of Eqs. (18-21) reveals that, if $m_i - m_j \not\equiv 0$ (mod 2) and $m_i + m_j \not\equiv 0$ (mod 2),

$$Q_{m_i m_j} = R_{m_i m_j} = 0 \tag{22}$$

This simple conclusion leads to the uncoupling of the buckling Eqs. (7) into two sets: one gives the symmetric longitudinal buckling modes, and the other gives the antisymmetric longitudinal buckling modes. The equations derived here involve no summations, the computation is thus greatly simplified.

Further Consideration of Identical and Equally Spaced Discrete Stringers and Rings

A typical stiffness matrix element in the buckling Eq. (7) may be written in the symbolic form

$$C(i,j) = \delta_{ij}A + \delta_{m_i m_j}(BN_{n_i n_j} + DM_{n_i n_j}) + \delta_{n_i n_j}(EQ_{m_i m_j} + FR_{m_i m_j}) \tag{23}$$

Equation (23) indicates that the odd numbers of m_i and the even numbers of m_j are not related. In other words, if a row in the buckling matrix corresponds to some odd number of m_i, then all of the elements in that row that correspond to the even numbers of m_j are zero. For a sample case with two stringers ($n_s = 2$), the nonzero elements in the matrix are shown in Fig. 2a. By properly interchanging rows and columns in Fig. 2a, the uncoupled version of matrix is obtained and shown in Fig. 2b. Figure 2b shows that the submatrix that corresponds to the odd numbers of m_i and m_j results in a symmetric longitudinal buckling mode, whereas the other submatrix that corresponds to the even numbers of m_i and m_j results in an antisymmetric longitudinal buckling mode.

Extending this idea to the circumferential buckling mode, Eqs. (14) can be reduced as follows

$$M_{n_i n_j} = N_{n_i n_j} = 0 \quad \text{if} \quad \begin{cases} n_i - n_j \not\equiv 0(\text{mod } n_s) \\ n_i + n_j \not\equiv 0(\text{mod } n_s) \end{cases} \tag{24}$$

Equations (24) indicate that, if some row in the buckling matrix corresponds to n_i, then all the elements in that row that correspond to the number n_j which satisfy $n_i - n_j \not\equiv 0$ (mod n_s) and $n_i + n_j \not\equiv 0$ (mod n_s) are zero. In other words, the elements with n_i and n_j that satisfy $n_i - n_j \equiv 0$ (mod n_s) and $n_i + n_j \equiv 0$ (mod n_s) are not zero. For example, if $n_s = 8$, the elements that correspond to the combinations with (n_i, n_j = 1, 7, 9, 15, 17,...), (n_i, n_j = 2, 6, 10, 14,...), (n_i, n_j = 3, 5, 11, 13, 19, 21,...), (n_i, n_j = 4, 12, 20, 28, ...), and (n_i, n_j = 0, 8, 16, 24, 32,...) have nonzero values. These relations may be illustrated in Figs. 2a, 2b, and 2c for the case of $n_s = 2$.

These procedures can be summarized as follows. The whole set of buckling equations is first uncoupled into two sets: one related to the symmetric circumferential modes and the other related to the antisymmetric circumferential modes. Each of these two sets is then uncoupled into two subsets: one related to the symmetric longitudinal modes and the other related to the antisymmetric longitudinal modes. The lowest eigenvalue that comes from these four subsets is the buckling load of the whole system.

Numerical Method for Buckling Equations

As shown in Eq. (12), for the case of equally spaced and identical stringers, the buckling Eqs. (7) can be reduced into two sets; one related to the antisymmetric circumferential modes and the other related to the symmetric circumferential modes. Both sets, although different in content, can be described in the same symbolic form.

$$\left[\begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} \end{bmatrix} - \begin{bmatrix} \tilde{0} & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{D}_{33} \end{bmatrix}\right] \begin{Bmatrix} \tilde{U} \\ \tilde{V} \\ \tilde{W} \end{Bmatrix} = \begin{Bmatrix} \tilde{0} \\ \tilde{0} \\ \tilde{0} \end{Bmatrix} \tag{25}$$

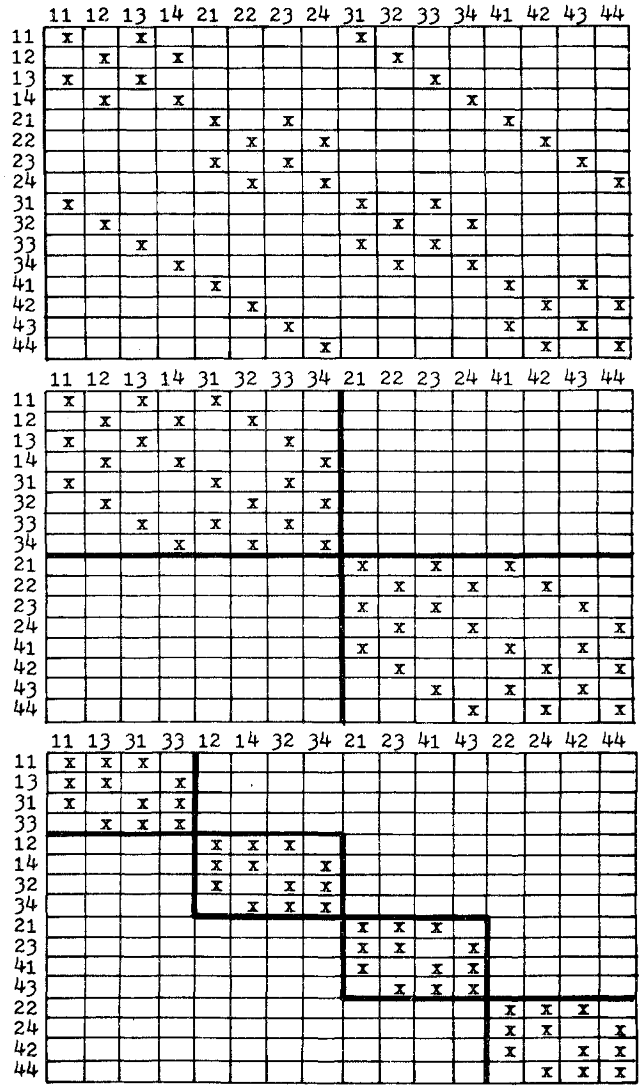

Fig. 2 a) Nonzero elements in a typical buckling matrix; b) two uncoupled submatrices with symmetric and antisymmetric longitudinal modes; c) four further uncoupled submatrices with circumferentially related modes.

In order to save computer storage and computing time by a special compact storage scheme,[18] effort to preserve the original form of the sparseness of the buckling submatrices is made.

Through some manipulations, Eqs. (25) are reduced to

$$\tilde{B}\tilde{W}=\lambda\tilde{A}\tilde{W} \tag{26}$$

where

$$\tilde{A}=\tilde{C}_{33}-\tilde{C}_{31}\tilde{C}_{11}{}^{-1}\tilde{C}_{12}\tilde{E}^{-1}\tilde{C}_{21}\tilde{C}_{11}{}^{-1}\tilde{C}_{13} + \tilde{C}_{31}\tilde{C}_{11}{}^{-1}\tilde{C}_{12}E^{-1}\tilde{C}_{23}-\tilde{C}_{31}\tilde{C}_{11}{}^{-1}\tilde{C}_{13} + \tilde{C}_{32}E^{-1}\tilde{C}_{21}\tilde{C}_{11}{}^{-1}\tilde{C}_{13}-\tilde{C}_{32}E^{-1}\tilde{C}_{23} \tag{27a}$$

$$\tilde{B}=\lambda\tilde{D}_{33} \tag{27b}$$

$$\tilde{E}=\tilde{C}_{22}-\tilde{C}_{21}\tilde{C}_{11}{}^{-1}\tilde{C}_{12} \tag{27c}$$

For uniform axial compression and external pressure, matrices $\tilde{A}$ and $\tilde{B}$ are both positive definite and symmetric. Because only the smallest eigenvalue λ is of interest, the Power method may be used. Unfortunately, for the present class of problems the eigenvalues are poorly separated and the convergence is slow.

An effective algorithm called Ritz iteration was developed by Rutishauser[19] for standard eigenvalue problems. It can be extended to solve the eigenvalue Eqs. (26) via the matrix decomposition of $\tilde{A}$. The convergence rate of this method is considerably improved over the Power method. It is, however, still slow for the present problem because quite a few eigenvalues are very close to the minimum eigenvalue. Rutishauser[19] also suggested that if Chebyshev iteration is used in addition to Ritz iteration, considerable improvement in convergence rate can be achieved, especially when convergence is slow. This procedure was successfully applied to the present large eigenvalue equations which contain poorly separated eigenvalues. This method can also calculate the equal eigenvalues which may appear in the vibration problem.

The convergence characteristics of the Power method, the Ritz iteration method, and the Ritz iteration method combined with Chebyshev procedure were studied through a numerical example. The example was derived from Eqs. (25) with $k_m=k_m'=4$ and $k_n=k_n'=8$. The matrix given by Eqs. (25) had an order of 96 which was reduced to 32 by using Eqs. (26). The results are summarized in Table 1. The Ritz iteration method with Chebyshev procedure appears to be most efficient among the methods compared. This method was used in the example studies.

Results

Simply Supported Cylinder with Many Stiffeners under Uniform Axial Compression by Both Smeared-Out and Discrete Techniques

The first example was defined by the dimensions and properties that $E=1.06\times10^7$ psi; $\nu=1/3$; $a_1=a_2=3$ in.; $d_1=d_2=0.2$ in.; $t_1=t_2=0.125$ in; $h=0.05$ in.; $r=48$ in.; $L=50$ in.; and weight = 100 lb. The results are presented in Table 2.

In the discrete calculations, 16 rings and 100 stringers were used. The first discrete calculation was based on the displacement functions with 600 terms ($k_m=k_m'=10$, $k_n=k_n'=20$), an antisymmetric circumferential mode, and uniform strain distribution. The second discrete calculation was based on five antisymmetric circumferentially related modes and 24 symmetric longitudinal modes ($k_n=5$ with $n=17$, 83, 117, 183, 217; $k_m=24$ with $m=1$, 3, 5,...47) and uniform strain distribution. The smeared-out calculation was based on a uniform stress assumption and the result agreed with that obtained previously.[10] Table 2 serves to verify the correctness of the present discrete formulations and solution methods. Because the torsional stiffnesses of the stiffeners were neglected in the smeared-out calculation, the resulted buckling load should be slightly smaller than those obtained by the discrete calculations.

When the numbers of rings and stringers are large, the buckling loads obtained by using uniform strain and uniform stress assumptions should be close. The first discrete computation was also performed by using the uniform stress assumption; a buckling load of 245.7 kips was obtained. Since the uniform strain assumption introduces circumferential tensions in the rings which produce compressive stresses in the skin, it always yields lower buckling load than the uniform stress assumption.

If the stiffeners in this example are neglected, the buckling stress σ_x for the monocoque cylinder becomes 6750 psi. The tremendous strengthening effect due to a large number of stiffeners is seen.

Simply Supported Cylinder with One Ring and Four Stringers under Uniform Axial Compression

The second example was defined by the dimensions and properties that $E=1.06\times10^7$ psi; $\nu=1/3$; $h=0.03$ in.; $r=48$ in.; $L=50$ in.; $d_s=d_r=1.3196$ in.; $t_s=t_r=0.8248$ in.; $Z=1637$; weight = 100 lb. The buckling stresses computed by using the uniform stress assumption and either symmetric or antisymmetric circumferential mode are given in Table 3. The converged buckling stress is 4070 psi. Table 3 reveals that one should not be prejudiced against the u and v functions by using more terms in the w function.

Table 1 Comparison of convergence

Method	Mode number Lowest	2nd	3rd	4th	CDC 6500 CP time, sec	Iteration
Exact	1511.627	1544.382	1590.751	1772.99		
Power	1511.630				12.0	245
Ritz iteration	1511.629	1544.393	1590.759	1797.46	7.3	30×4
Ritz & Chebyshev	1511.627	1544.382	1590.751	1778.94	4.5	22×4

Table 2 Uniform axial buckling load P_x for example 1

Method	P_x, kips	$P_x/2\pi r$, lb/in.	σ_x, psi	Mode numbers m	n
1) Discrete	243.7	808.0	13,891	6	17
2) Discrete	241.7	801.3	13,776	6	17
3) Smeared-out	238.7	791.5	13,597	6	17

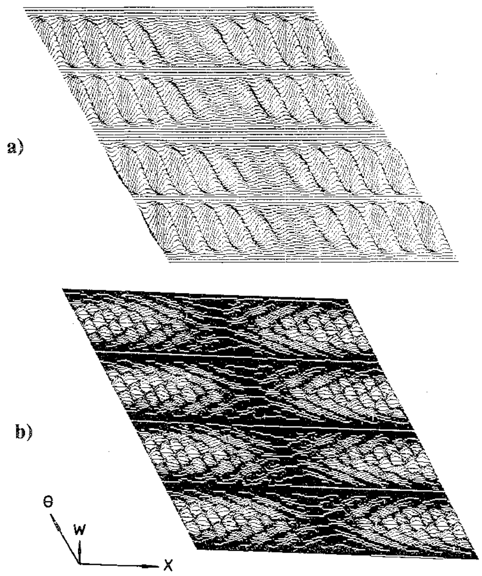

Fig. 3 Buckling mode shapes for antisymmetric circumferential mode and a) mixed longitudinal modes ($\sigma_x = 4070$ psi; $k_m = 28$; $k_n = 6$; $n = 1, 2, ...6$); b) symmetric longitudinal mode ($\sigma_x = 4068$ psi; $k_m = 15$; $k_n = 7$; $n = 4, 8, ...28$).

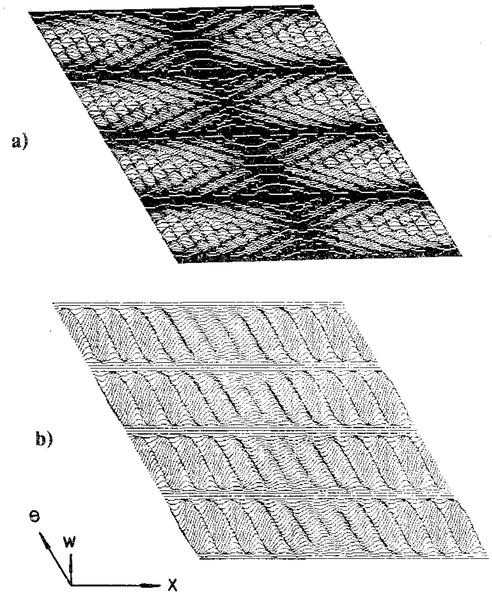

Fig. 4 Buckling mode shapes for antisymmetric longitudinal mode and a) symmetric circumferential mode ($\sigma_x = 4048.4$ psi; $k_m = 15$; $k_n = 7$; $n = 2, 6, ...26$); b) antisymmetric circumferential mode ($\sigma_x = 4068.1$ psi; $k_m = 14$; $k_n = 6$; $n = 2, 6, ...22$).

The buckling calculation was then performed by assuming uniform stress and either symmetric or antisymmetric circumferential mode, either symmetric ($m = 1, 3, ... 2k_m + 1$) or antisymmetric ($m = 2, 4, ... 2k_m$) longitudinal modes, and circumferentially related modes. The results for buckling stresses and mode shapes are shown in Figs. 3a, 3b, 4a, and 4b, respectively. The mode shapes were plotted by the Gould 4800 electrostatic printer. The four mode shapes in the figures are quite different from one another, but the buckling stresses are almost identical ($\sigma_x = 4068$ psi). This buckling stress is very close to that for the same cylinder without stiffeners ($\sigma_x = 4047$ psi; $P_x = 36{,}613$ lb; $m = 2$; $n = 3$). The results indicate that a small number of discrete stiffeners is not effective in raising the buckling strength ($\sigma_x = 4047$ psi) for the same monocoque cylinder even if the mode shapes are drastically changed. In the case of a flat plate, however, a discrete stiffening increases the buckling strength substantially.

The buckling load for this problem was also calculated based on the uniform strain assumption. The antisymmetric

Table 3 Buckling stresses for the discretely stiffened cylinder

k_m	k_n	k'_m	k'_n	No. of terms	Buckling stress, σ_x, psi	Relative error	Iteration no.	CDC-6500 CP time, sec
					Antisymmetric circumferential mode			
12	2	12	2	72	10,107.0	3.6×10^{-8}	10×4	4
14	2	14	2	84	8165.6	8.2×10^{-8}	12×4	5
13	3	13	3	117	9498.3	1.8×10^{-7}	22×4	13
12	4	12	4	144	10,107.0	6.5×10^{-9}	19×4	19
24	2	24	2	144	5581.3	3.4×10^{-8}	33×4	25
26	2	26	2	156	5576.0	1.6×10^{-7}	50×4	36
28	2	28	2	168	5575.0	6.3×10^{-7}	51×4	43
30	2	30	2	180	5574.6	3.4×10^{-7}	58×4	54
24	4	24	4	288	5581.3	9.9×10^{-7}	79×4	288
10	10	10	10	300	11,319.0	5.3×10^{-11}	38×4	142
10	11	10	11	330	10,916.0	1.0×10^{-7}	30×4	–
11	10	11	10	330	9910.3	4.2×10^{-8}	27×4	156
28	8	14	4	336	4320.3	4.7×10^{-7}	115×5	992
9	13	9	13	351	11,798.0	6.5×10^{-7}	52×4	239
25	5	25	5	375	4070.3	2.0×10^{-8}	43×4	234
26	6	26	6	468	4070.3	4.5×10^{-8}	59×4	536
28	6	28	6	504	4070.0	9.9×10^{-7}	84×4	632
26	7	26	7	546	4071.6	3.7×10^{-5}	40×4	582
27	7	27	7	567	4070.3	7.9×10^{-7}	114×4	964
26	8	26	8	624	4070.6	2.0×10^{-5}	35×4	901
					Symmetric circumferential mode			
12	2	12	2	71	17,295.0	4.8×10^{-7}	19×4	5
14	2	14	2	83	17,295.0	6.5×10^{-7}	17×4	6
13	3	13	3	116	16,808.0	2.3×10^{-7}	25×4	14
12	4	12	4	143	11,073.0	2.2×10^{-7}	8×4	15
24	4	24	4	287	5582.3	4.2×10^{-9}	33×4	114
25	5	25	5	373	4070.3	3.9×10^{-9}	39×4	271
10	17	10	17	539	8383.0	1.9×10^{-7}	27×4	457

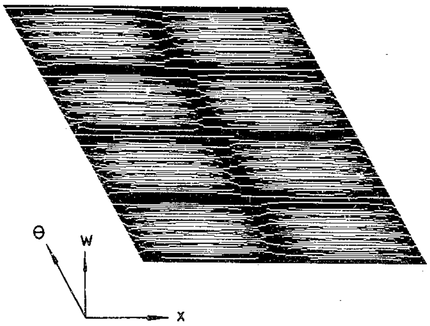

Fig. 5 Buckling mode shape for simply supported cylinder with one ring and four stringers under axial compression (uniform strain assumption).

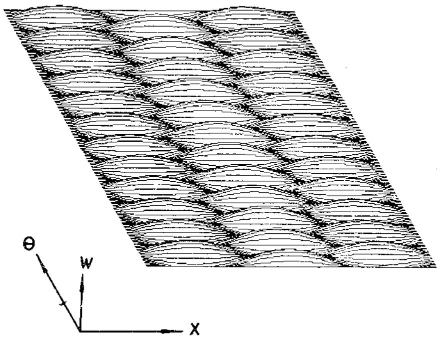

Fig. 6 Buckling mode shape for clamped cylinder with two inner rings and eight outer stringers under external pressure.

circumferential mode, the antisymmetric longitudinal modes ($m=2$, 4,...32), and the circumferentially related modes ($n=2$, 6,...30) governed the buckling equations. The buckling values obtained were $P_x=21{,}632$ lb; $\sigma_x=1639$ psi; $\sigma_\theta=229.5$ psi. The mode shape is shown in Fig. 5. It is of interest to note that this buckling load is even less than that (36,613 lb) for the same cylinder without stiffeners. This difference is caused by the basic assumptions. In the case of uniform stress assumption, there is no circumferential stress. But in the case of uniform strain assumption, the ring introduces high compressive circumferential stress in the skin that destabilizes the shell. It is noted that the computing time and storage were drastically reduced in treating the submatrices of smaller sizes in Eqs. (26) and (27).

C. Comparison with Experimental Results

Esslinger and Geier[20] tested a series of discretely stiffened Mylar cylinders with edges cast into rigid end plates. The modulus of elasticity of Mylar, which is not available in Ref. 20, has a wide range. However, based on the geometric parameter Z of 5320, the Poisson's ratio of ⅓, and the theoretical buckling load of 78 kg given in Ref. 20 for a monocoque cylinder, the modulus of elasticity was obtained as 553.6 kg/mm^2 by using the classical formula that the Euler buckling stress is $Eh/[3(1-\nu^2)]^{1/2}r$. Two very stiff rings were placed at the ends of the cylinder to simulate the built-in edge conditions that $w=v=\partial w/\partial x=N_x=0$.

Cylinder with Eight Outer Stringers and Two Inner Rings under External Pressure

The geometry of the cylinder was defined as: $L=330$ mm; $r=100$ mm; $h=0.193$ mm. The stiffeners were T-type with two back-to-back angle bars of $6\times6\times0.193$ mm. The buckling computations were based on the antisymmetric circumferential mode, the symmetric longitudinal modes ($m=1$,3,...41), and the circumferentially related modes ($n=4$, 12, 20,...52). For the uniform strain and uniform stress assumptions, the buckling pressures obtained were 0.009768 and 0.01002 kg/cm^2, respectively. Both are in close agreement with the experimental value[20] of 0.0095 kg/cm^2. The buckling mode shape was plotted in Fig. 6 with $m=3$ and $n=12$. The mode shape and mode numbers agree well with those shown in the photograph (Fig. 11) of Ref. 20. For the same cylinder without stiffeners the buckling pressure is only about 0.0039 kg/cm^2.[20]

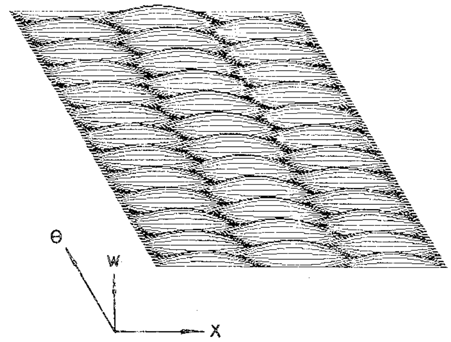

Fig. 7 Buckling mode shape for clamped cylinder with two inner rings and eight outer stringers under axial compression.

Cylinder with Eight Outer Stringers and Two Inner Rings under Uniform Axial Compression

The cylindrical shell considered was the same as the previous one, but all of the stiffeners were single angle bars of $6\times6\times0.193$ mm.

The stiffeners were first neglected. In the buckling computation, the antisymmetric circumferential mode with $n=6$ and the symmetric longitudinal modes with $m=1$, 3...79 were used. The buckling load obtained was 78.2 kg which agreed well with the theoretical value of 78 kg given in Ref. 20. The buckling mode shape found (with one-half longitudinal sine wave) also agreed with the experimental photograph.[20]

As to the buckling load of the stiffened shell, Ref. 20 provided an experimental value of 45 kg, a theoretical value of 80 kg, and an experimental postbuckling value greater than 95 kg. The present computations were based on the same modes as those used in the case of external pressure. Based on the uniform strain assumption, the buckling load obtained was 75.7 kg. The mode shape is shown in Fig. 7. The 75.7 kg predicted is close to the theoretical value of 80 kg given in Ref. 20. The present buckling mode shape agrees well with that obtained experimentally in Ref. 20. It is noted that the lack of agreement between theoretical and experimental buckling loads in the case of axially compressed monocoque cylindrical shell is well known. This phenomenon also appears in the present case of discretely stiffened cylindrical shell.

A buckling load of 92.0 kg was obtained when the uniform stress assumption was used. This value should be expected to be higher than that (75.7 kg) obtained by using the uniform strain assumption. The reason lies mainly in the difference in circumferential stress. The circumferential stress is zero in the case of uniform stress, while in the case of uniform strain, it is 2.2% of the longitudinal stress which corresponds to 90.8% of the circumferential buckling stress for the same cylinder without stiffeners under external pressure. This circumferential stress in the shell is introduced by the two rings in circumferential tension.

Concluding Remarks

The buckling equations for the cylindrical shell with multiple stiffener sizes are formulated using the energy method. For the case of equally spaced and identical stringers and equally spaced and identical rings, a close examination of the buckling equations leads to the uncoupling of the buckling equations into separate sets of equations associated with the symmetric and antisymmetric longitudinal modes, symmetric and antisymmetric circumferential modes, and circumferentially related modes. By using such uncoupling and some matrix partitioning and reduction, the huge sets of matrix equations required for the buckling predictions of a common discretely stiffened cylinder can be reduced drastically to manageable sizes. Effort is made to preserve the sparseness of the buckling matrices so that a special compact storage technique can be used.

A method, the Ritz iteration with Chebyshev procedure, is developed for efficient computation of the minimum eigenvalue of a large general eigenvalue problem. The superiority of this method over other methods is demonstrated through a numerical example.

The discrete formulations are first used to analyze an axially compressed cylinder with a large number of stiffeners for which an alternative solution based on the smeared-out method can be obtained for comparison. The agreements are very good for the formulations based on both the uniform stress and uniform strain assumptions (see Table 2). The formulations are then verified by comparing the computed buckling loads and modes with the experimental results[20] for a clamped cylinder with two rings and eight stringers under external pressure and then under axial compression. For the case of external pressure, the agreements are good for the formulations based on both the uniform stress and uniform strain assumptions. For the case of axial compression, the agreements are poor. Such lack of agreement between theoretical and experimental axial buckling loads has long existed in the case of the monocoque cylinder.

It should be noted that both the uniform stress and uniform strain assumptions are shown to be consistent with each other and correct for the cylinder under external pressure. This is the important case where a small number of discrete stiffeners is found to be very effective in raising the buckling loads. Such conclusion was made previously by Esslinger and Geier in an experimental study.[20] For the case of axial compression, the two prestress assumptions do not result in the same buckling load. The uniform strain assumption results in circumferential compression in the skin and consequently, results in lower buckling load than the uniform stress assumption. However, this is the unimportant case where a small number of discrete stiffeners is found to be ineffective in raising the buckling loads.

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